Plan

1. Phonon Polaritons
2. Exciton Polaritons
3. Non-linear optics phenomena

From last time:

When diagonalizing H, we must solve the generalized Eigenvalue problem

$$H\Psi = \varepsilon \Sigma \Psi$$

where $\Sigma = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

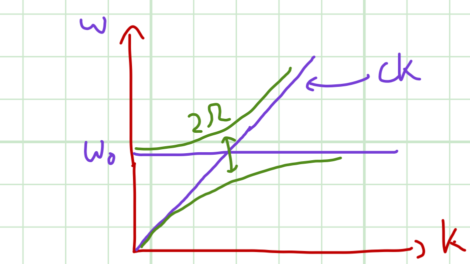
$$\begin{array}{c|cccc} & a_k & c_k & a_{-k}^+ & c_{-k}^+ \\ \hline c_k^+ & \omega_k & -i\Omega & 0 & -i\Omega \\ c_k^+ & i\Omega & \omega_0 & -i\Omega & 0 \\ a_{-k} & 0 & i\Omega & \omega_k & i\Omega \\ c_{-k} & i\Omega & 0 & -i\Omega & \omega_0 \end{array} \begin{vmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{vmatrix} = \omega \begin{vmatrix} \alpha \\ \beta \\ -\gamma \\ -\delta \end{vmatrix} \Bigg\} \Rightarrow$$

these signs are needed to ensure commutation relations between the new operators are satisfied.

$$\Rightarrow \mathrm{Det}\left[\mathcal{H} - \begin{pmatrix} \omega & & & \\ & \omega & & \\ & & -\omega & \\ & & & -\omega \end{pmatrix}\right] = \omega^4 - \omega^2(\omega_0^2 + \omega_k^2) + \omega_0^2\omega_k^2 - 4\Omega^2\omega_0\omega_k = 0$$

$\Rightarrow$ I think OS. has an error in equation for ω, p. 351. $-4\Omega^2\omega_k^2 \Rightarrow -4\Omega^2\omega_k\omega_0$

Plot the roots:



$$\Omega \to 0 \Rightarrow \omega^4 - \omega^2(\omega_0^2 + \omega_k^2) + \omega_0^2\omega_k^2 = 0$$

$$\omega^2 = \frac{\omega_0^2 + \omega_k^2 \pm \sqrt{(\omega_0^2 + \omega_k^2)^2 - 4\omega_0^2\omega_k^2}}{2}$$

$$= \frac{\omega_0^2 + \omega_k^2 \pm |\omega_0^2 - \omega_k^2|}{2}$$

$$\omega = \{\omega_0, \omega_k\}$$

$$k \to 0 \Rightarrow \omega^4 - \omega_0^2\omega^2 = 0$$

$$\omega = \{0, \omega_0\}$$

$\frac{\Omega}{\omega_0}$ small: $\omega \to \omega_0 \pm \Omega$

$ck = \omega_0$ $\quad \Psi = \frac{1}{\sqrt{2}}(a_k \pm i c_k)$

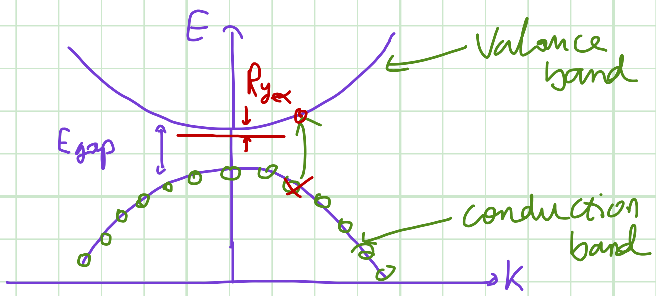
Exciton-polaritons

Instead of optical phonons we have excitons ⇒ bound p-h pairs

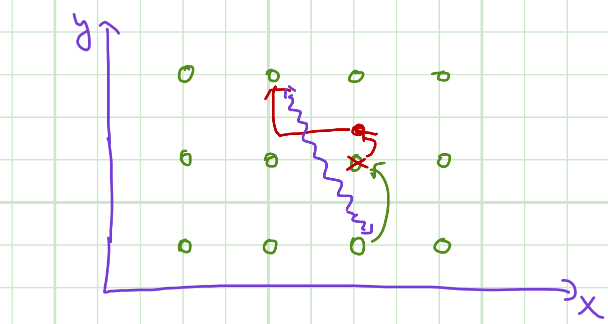
⇒ what are these p-h pairs?

⇒ consider a semiconductor

Momentum space picture



Real space picture



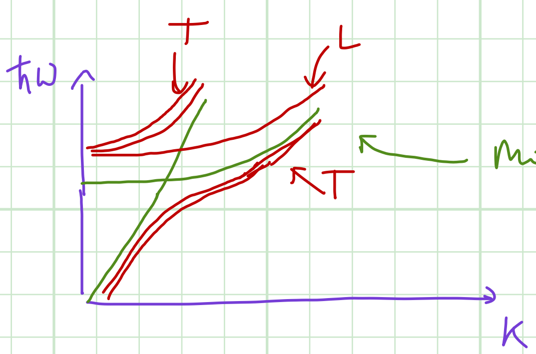
$$E_x = E_{gap} - Ry_{ex} + \frac{\hbar^2 k^2}{2m_{ex}}$$

COM KE

gap between V + C bands

binding Energy

⇒ to proceed, we substitute E_x for $\hbar\omega_0$ in the phonon-polariton expressions



main modification is spectrum becomes curved at high K.

Non-linear optics

So far we have assumed that

$$P = \epsilon_0 \chi \vec{E}$$

⇒ This can be thought of as the first term in a power series expansion

$$P = \epsilon_0 \chi E + 2\chi^{(2)} E^2 + 4\chi^{(3)} E^3 + \ldots$$

Where are these coeff. coming from?

Higher order terms naturally occur if we couple to a non-linear oscillator, or if we go to higher orders for the two level system...

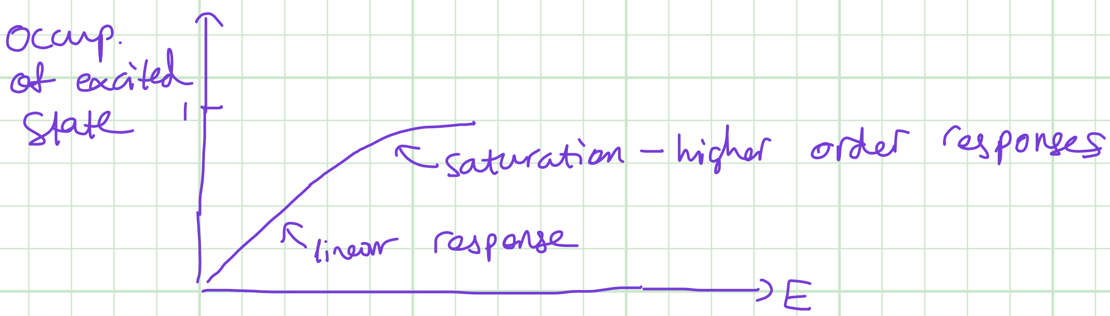
Example: beyond linear response in a 2-level system

Remember that the perturbation $V_{int}(t') = ME \cdot E(t')$

$$\rho(t) = \left\langle \left(1 + i\int_{-\infty}^{t} dt'\, V_{int}(t') - \frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t'} dt'\, dt''\, V_{int}(t') V_{int}(t'')\right) \rho(t) \left(1 - i\int_{-\infty}^{t} dt'\, V_{int}(t') - \frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t'} dt'\, dt''\, V_{int}(t') V_{int}(t'')\right)\right\rangle$$

$$= \underbrace{i\int_{-\infty}^{t} dt' \left[V_{int}(t')\rho(t) - \rho(t) V_{int}(t')\right]}_{\text{linear response}} + \underbrace{\int_{-\infty}^{t}\int_{-\infty}^{t} dt' dt'' \langle V_{int}(t')\rho(t) V_{int}(t'')\rangle - \frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t'} dt' dt'' \langle V_{int}(t') V_{int}(t'')\rho(t) + \rho(t) V_{int}(t') V_{int}(t'')\rangle}_{\text{quadratic response } \chi^{(2)}}$$

$+ \ldots$



2^{nd} Harmonic generation + 3-wave mixing

Start with Maxwell's wave equation with sources but $\epsilon \equiv \epsilon_0$

$\Rightarrow$ The usual Maxwell's wave eq. in medium only designed to work for linear response, so we will need to derive a more general eq.

$$\frac{\partial^2 E}{\partial x^2} = \epsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2} + \mu_0 \frac{\partial J}{\partial t}$$

ignore polarization

$\Leftarrow$ from last time

$\Box \vec{A} = \vec{j}$ take time derivative of both sides and use $E = \frac{\partial A}{\partial t}$

set $\varphi = 0$

J can be related to P

$-\,0$

$\Uparrow J$

$+\,0$

$$J \equiv \frac{\partial P}{\partial t} = \frac{\partial}{\partial t}\left[\epsilon_0 \chi E + 2\chi^{(2)} E^2 + \ldots\right]$$

stop here for now

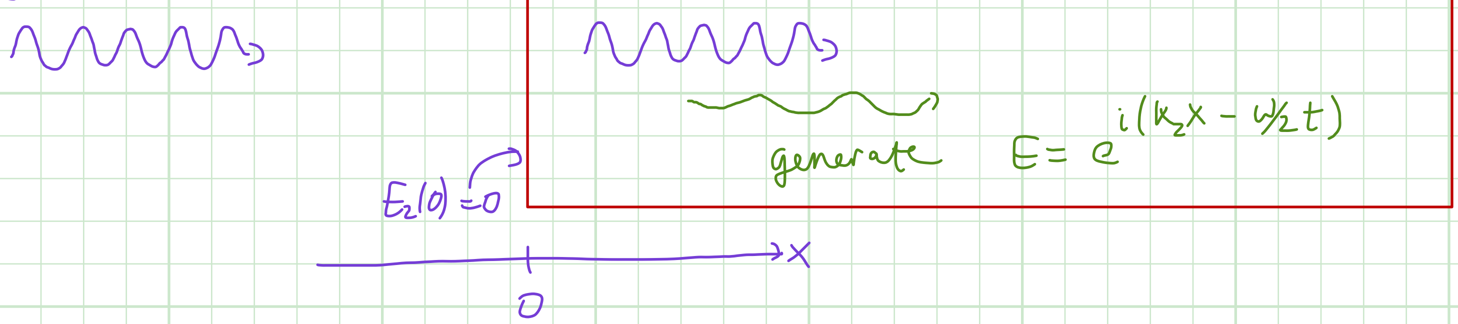
Plugging into the wave equation, we get

$$\frac{\partial^2 E}{\partial x^2} = \epsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2} + \mu_0 \frac{\partial^2}{\partial t^2}\left(\epsilon_0 \chi E + 2\chi^{(2)} E^2 + \ldots\right) \qquad \frac{\partial^2}{\partial t^2} E^2 = \frac{\partial}{\partial t} 2E\dot{E} = 2E\ddot{E} + 2\dot{E}\dot{E}$$

$$\frac{\partial^2 E}{\partial t^2} = \underbrace{\epsilon_0\mu_0(1+\chi)\frac{\partial^2 E}{\partial t^2}}_{\text{usual wave eq in medium}} + \underbrace{2\mu_0\chi^{(2)}\left[2E\ddot{E} + 2(\dot{E})^2\right]}_{\text{non-linear part}}$$

Non-linear part induces mode coupling

Send in $E = e^{i(kx-\omega t)}$



Let's see how this works

$$E(x,t) = E_0(x)\, e^{i(kx-\omega t)} + E_2(x)\, e^{i(k_2x-\omega_2 t)} \qquad \omega/2 = \omega_2$$

Add the assumption that $\chi^{(2)}$ is small, so 2^{nd} harmonic is very gradually generated as a function of position, and $E_0 \gg E_2$

$$\frac{\partial^2 E}{\partial x^2} = \frac{\partial}{\partial x}\left[E_0'(x)\, e^{i(kx-\omega t)} + ik E_0(x)\, e^{i(kx-\omega t)} + E_2'(x)\, e^{i(k_2x-\omega_2 t)} + ik_2 E_2(x)\, e^{i(k_2x-\omega_2 t)}\right]$$

$$= 2ik\, E_0'(x)\, e^{i(kx-\omega t)} \underline{\underline{- k^2 E_0(x)\, e^{i(kx-\omega t)}}}_{①} + 2ik_2\, E_2'(x)\, e^{i(k_2x-\omega_2 t)} \underline{\underline{- k_2 E_2(x)\, e^{i(k_2x-\omega_2 t)}}}_{②}\Big]$$

$$= \epsilon\mu_0 \frac{\partial^2 E}{\partial t^2} + 4\mu_0\chi^{(2)}\left(E\ddot{E} + \dot{E}^2\right)$$

$$= \underline{\underline{-\epsilon\mu_0\,\omega^2 E_0\, e^{i(kx-\omega t)}}}_{①} \underline{\underline{- \epsilon\mu_0\omega_2^2 E_2\, e^{i(k_2x-\omega_2 t)}}}_{②} + 4\mu_0\chi^{(2)}\left[E_0^2(-2\omega^2)\, e^{2i(kx-\omega t)}\right]$$

Remaining terms: $\cancel{2}ik_2\, E_2'(x)\, e^{i(k_2x-\cancel{\omega_2 t})} = 4\mu_0\chi^{(2)}E_0^2(-\cancel{2}\omega^2)\, e^{2i(kx-\cancel{\omega t})}$

$$E_2'(x) = i4\mu_0\chi^{(2)}\frac{E_0^2\omega^2}{k_2}\, e^{i(k_2-2k)x} = 4i\mu_0\frac{c}{n}\chi^{(2)}E_0^2\,\omega\, e^{i(k_2-2k)x}$$

$$k_2 = \frac{n\omega_2}{c}$$

$$= \frac{4i\mu_0}{\sqrt{\mu_0\epsilon}}\chi^{(2)}E_0^2\,\omega\, e^{i(k_2-2k)x}$$

$$= 4i\sqrt{\frac{\mu_0}{\epsilon}}\,\chi^{(2)}E^2\omega\, e^{i(k_2-2k)x}$$

Assuming $E_2(x=0) = 0$, $2k - k_2 = \Delta k$

$$E_2(x) = 4\sqrt{\frac{\mu_0}{\epsilon}}\,\chi^{(2)}E^2\omega\left[\frac{e^{i\Delta k x}-1}{\Delta k}\right] = 4\sqrt{\frac{\mu_0}{\epsilon}}\,\chi^{(2)}E^2\left(\frac{x}{2}\right)e^{i\Delta k\cdot x/2}\left[\frac{e^{i\Delta k\cdot x/2}-e^{i\Delta k\cdot x/2}}{(\Delta k\cdot x/2)}\right]$$

$$\Rightarrow |E_2(x)|^2 = 16\frac{\mu_0}{\epsilon}[\chi^{(2)}]^2E^4\omega^2x^2\left[\frac{\sin^2(\Delta k x/2)}{(\Delta k x/2)^2}\right]$$

So if $\Delta k \to 0$ [phase match]

intensity of frequency doubled beam $\sim x^2 \Rightarrow$ length of propagation

$\Rightarrow$ standard practice for laser systems

$\Rightarrow$ Green laser pointer

AlGaAs diode

$\hookrightarrow$ 808 nm Nd:YVO$_4$ laser $\xrightarrow{1064\,nm}$ KTP crystal $\rightarrow$ 532 nm [green light]

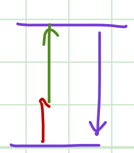
$\Rightarrow$ "phase matching" the condition $\Delta k = 0$ is typically satisfied only along certain directions $\Rightarrow$ These should be chosen for best output intensity

$\Rightarrow$ Generally $\chi^{(2)}$ non-linearity can "mix" 3 frequencies

$\omega_1 \pm \omega_2 \pm \omega_3 = 0 \Rightarrow$ "3-wave mixing"

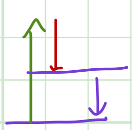
$\Rightarrow$ this is also commonly used in optics

Sum freq generator



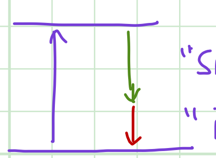
in out

diff. freq. generator



in out

Parametric down conv.



in out

Alternative way to think about non-linearity

$$n(E) = \sqrt{1+\chi+2\frac{\chi^{(2)}}{\epsilon_0}E+\dots}$$

This is like a difraction grating with $\Delta n \sim E$

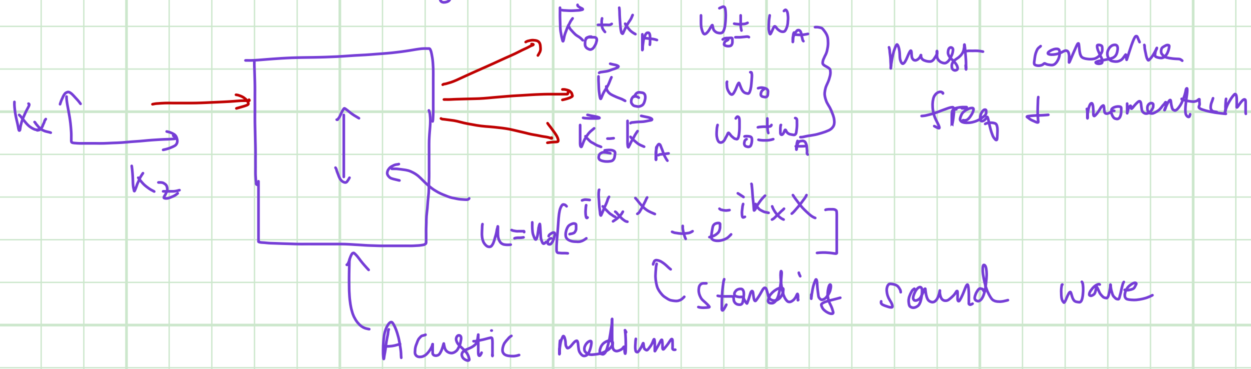
E.g. main beam(s) create a difraction grating for the other beams

Acusto-optics + the AOM [Acusto-optical modulator]

Instead of using light to make a diffraction grating [or to give a kick]
We can use ultrasound

$\Rightarrow$ χ depend on strain u, which comes from the sound waves

$$\chi = \chi + \frac{\partial \chi}{\partial u} u$$



other outputs?

The above device is called an AOM or Bragg cell

$\Rightarrow$ used in telecom + in optics / cold atoms experiments

$\Rightarrow$ fast control of freq. shift + output intensity

$\Rightarrow$ mode locking, imaging systems for UCA, lattice systems for UCA

intensity + phase shift.

$\Rightarrow$ conveyor belt

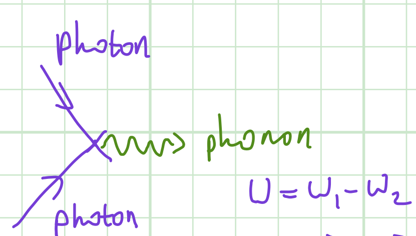
Raman scattering: AOM in reverse

$\Rightarrow$ Hard to directly couple photons to phonons due to large mismatch in freq.

$\Rightarrow$ use a 2-photon process

$$H' = P \cdot E = \text{linear part} + \epsilon_0 u E^2 \frac{\partial \chi}{\partial u}$$

$$a_K \, a_q \, c^{\dagger}_{K+q}$$



$U = \omega_1 - \omega_2$

$K = \vec{k}_1 + \vec{k}_2$